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ON THE STABILITY OF LAMINAR FLAME

BY

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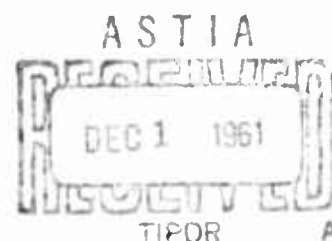
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F O R E W O R D

SUMMARY

The stability of a laminar flame subjected to a small perturba-
tion is examined. An important practical case is considered in which
the amplitude of the disturbance, as measured by the flame displace-
ment, is small in comparison with the wave length of the disturbance,
but not necessarily small in comparison with the flame thickness.
Starting from the basic equations governing a reactive mixture and
making use of an expansion procedure in the parameter $\mathcal{L}$, which is
defined as the ratio of the wave length to the flame thickness, the
following results are obtained:

1. For $\mathcal{L} \gg 1$, the phenomenological theory of Landau, in which
   the flame is idealized as a surface of discontinuity propa-
   gating with a constant speed, is justified. Accordingly,
   Landau's conclusion that a flame is unstable to small dis-
   turbances is strictly valid for disturbances of sufficiently
   long wave length.

2. The second term in the expansion in $1/\mathcal{L}$ yields a correc-
   tion to Landau's theory. Both the coefficent of thermal
   conduction and viscosity enter into the correction. The
   stabilizing influence of these additional terms is indicated;
   the details are reported elsewhere.

ON THE STABILITY OF LAMINAR FLAME

1. <u>Introduction</u>   There are two approaches to the study of the stability of laminar flame. In the first approach, the flame is treated as a surface of discontinuity endowed with certain properties and the stability of flame is examined from the hydrodynamic viewpoint without any reference to the detail chemical reactions involved. This is, therefore, a phenomenological approach. The principle contributions of this school are represented by the work of Landau (ref. 1) and Markstein (ref. 2). Landau considers the flame front as a surface of discontinuity which propagates with a constant speed, small in comparison with the local sound speed, and across which there is a discontinuous change in temperature. He found that such a surface is unstable to disturbances of all wave lengths. Markstein improved Landau's model by allowing the flame speed to depend on the flame curvature. Markstein found that the flame is stable with respect to disturbances of sufficiently small wave length.

The second approach to the study of flame stability takes into account the structure of the combustion zone and is certainly the more correct approach since it gives a detailed analysis of the changes which take place in the combustion front. The principle contributions of this school are represented by the work of Rosen (ref. 3) Barenblatt and Zeldovich (ref. 4) and Menkes (ref. 5). However, these authors have only examined the stability of flame relative to one dimensional disturbances.

In this paper we shall attempt to bridge the gap between the two approaches. If the stability of the phenomenological models of Landau and Markstein are truly relevant to the study of flame stability, it must be possible to justify these models from the first principles. In other words we must base our analysis on the fluid dynamic equations governing a reactive mixture and consider the problem of the stability of flame relative to three dimensional disturbances.

It is found that Landau's model can be justified provided that the wave length
of the disturbance is long enough, in comparison with the flame thickness, that
the changes across the flame are independent of the detailed profile of the
flame. Corrections to Landau's model for disturbances of moderate wave length
are also deduced. It is shown that the effect of heat conduction on flame
stability can indeed be described by a model similar to that used by Markstein. (cf., eq 13.9.

2. <u>Basic Equations</u> The composition of a mixture may be characterized con-
veniently by the mass-fractions Y_i (i = 1, 2, ... N) of the N species which
enter into the analysis. Clearly we have

$$\sum_{i=1}^{N} Y_i = 1 \tag{2.1}$$

We shall denote the desity of the ith-species by ρ_i and the density of the
mixture by ρ . It follows that for a gaseous mixture,

$$\rho = \sum_{i=1}^{N} \rho_i \tag{2.2}$$

and

$$Y_i = \rho_i / \rho \tag{2.3}$$

The theremodynamic state of a mixture is completely determined when the com-
position of the mixture and two of its independent thermodynamic state varia-
bles are given, e.g., when the pressure p, temperature T and the mass fractions
Y_1, Y_2, ... , Y_{n-1} are specified. Thus, the internal energy and enthalpy of
the mixture can be computed from

$$U = \sum_{i=1}^{N} Y_i U_i \tag{2.4}$$

$$H = \sum_{i=1}^{N} Y_i H_i \tag{2.5}$$

respectively. Certain other themodynamic relations describing chemical changes
are usually more conveniently expressed in terms of the mole-fraction X_i instead
of the mass-fraction Y_i. The two are, of course, related simply by

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$$X_i = \frac{Y_i/m_i}{\sum\limits_{i=1}^{N} Y_i/m_i}$$

$$\tag{2.6}$$

$$Y_i = \frac{m_i X_i}{\sum\limits_{i=1}^{N} m_i X_i}$$

$$\tag{2.7}$$

where m_i is the molecular weight of the ith species, since the molar concentration n_i is related to the mass density by

$$n_i = \rho_i/m_i \tag{2.8}$$

For a mixture of perfect gas, the equation of state is

$$p = nRT \tag{2.9}$$

where R is the universal gas constant and n is the total numbers of moles of all the species in the mixture, i.e., $n = \sum\limits_{i=1}^{N} n_i$.

The motion and changes of state of a chemically active mixture are governed by the following equations:

    i)   Continuity Equation

    ii)   Mass-Balance Equation for Each Species

    iii)   Equation of Motion

    iv)   Energy Equation

    v)   Equations of State

The derivation of these equations can be found elsewhere (e.g., ref. 6 or 7). The results are summarized below:-

Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0 \tag{2.10}$$

where $\vec{u}$ is the mass-average velocity of the mixture, t denotes time and ∇ is the del operator.

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Mass Balance Equation for Each Species

$$\rho \frac{DY_i}{Dt} + \nabla \cdot (\rho Y_i \vec{V}_i) = W_i , \qquad\qquad i = 1, 2, \dots, N \qquad (2.11)$$

where $\frac{D}{Dt}$ denotes the total derivative, W_i denotes the total rate of production of the ith species per unit volume and $\vec{V}_i$ is the diffusion velocity of the ith species. The diffusion velocities are related to the binary diffusion coefficients D_{ij} by the relation:

$$\sum_{j=1}^{N} \frac{1}{m_j D_{ij}} \left[\left(\vec{V}_i + D_{ij} \frac{\nabla Y_i}{Y_i} \right) - \left(\vec{V}_j + D_{ij} \frac{\nabla Y_j}{Y_j} \right) \right] = 0 , \qquad (2.12)$$
$$i = 1, 2, \dots N$$

Here we have ignored the pressure and thermal diffusion. Only $N-1$ of the above equations are independent, since the mass-average velocity $\vec{u}$ of the mixture is defined such that

$$\sum_{j=1}^{N} Y_j \vec{V}_j = 0 \qquad (2.13)$$

The conservation of mass demands that

$$\sum_{i=1}^{N} W_i = 0 \qquad (2.14)$$

Equation of Motion

$$\rho \frac{D\vec{u}}{Dt} = \nabla \cdot \bar{\bar{p}} \qquad (2.15)$$

where the stress tensor $\bar{\bar{p}}$ has the typical component

$$p_{jk} = -p \, \delta_{jk} + \mu \left[\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right) - \frac{2}{3} \delta_{jk} \frac{\partial u_\lambda}{\partial x_\lambda} \right] \qquad (2.16)$$

In (2.16), μ is the coefficient of viscosity of the mixture, δ_{jk} is the unit tensor and summation convention has been employed

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Energy Equation

$$\rho \frac{DH}{Dt} - \frac{Dp}{Dt} = - \nabla \cdot \vec{Q} + \Phi \tag{2.17}$$

where H is the enthalpy of the mixture given by (2.5), $\vec{Q}$ is the heat flux vector and Φ is the viscous dissipation:

$$\Phi = (p_{jk} - \delta_{jk} p) \frac{\partial u_j}{\partial x_k} \tag{2.18}$$

The heat flux vector $\vec{Q}$ consists of two parts: that due to conduction and that due to diffusion,

$$\vec{Q} = - \lambda \nabla T + \sum_{i=1}^{N} \rho Y_i \vec{V}_i H_i \tag{2.19}$$

where λ is the coefficient of thermal conduction and H_i is the enthalpy of the ith species.

Equation of State

In terms of the mass-fraction, the equation of state is

$$p = \rho R T \sum_{i=1}^{N} \frac{Y_i}{m_i} \tag{2.20}$$

3. **Chemical Reactions Involving Two Species**   Obviously the above system of equations is very complicated. To simplify somewhat the analysis, we shall limit our discussion to the special case of a single reaction involving two species: M_1 and M_2. We assume that they enter a reaction of the form

$$- \eta_1 M_1 \underset{k_b}{\overset{k_f}{\rightleftarrows}} \eta_2 M_2 \tag{3.1}$$

where k_f, k_b denote respectively the forward and backward specific reaction rate constants. Clearly, $- \eta_1 m_1 = \eta_2 m_2$. Eq. (3.1) includes two types of reactions of practical importance:

i) Unimolecular reversible reaction in which case

$$m_1 = m_2 \quad , \quad - \eta_1 = \eta_2 = 1$$

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ii) A one-stop decomposition reaction in which case

$$k_b = 0, \qquad \eta_1 = -1$$

In our case, $N = 2$ and it is convenient to introduce a new variable defined by

$$Y_2 = Y_{20} + m_2 \eta_2 \xi \tag{2.2a}$$

where Y_{20} is the initial (unburned) mass-fraction of the species M_2. It follows from (2.1) and the relation $-\eta_1 m_1 = \eta_2 m_2$ that

$$Y_1 = Y_{10} + m_1 \eta_1 \xi \tag{2.2b}$$

where Y_{10} is the initial mass fraction of the species M_1. ξ is, therefore, a measure of the composition of the mixture. We shall denote the rate of reaction (3.1) by $F(\xi, \rho, T)$ which is directly proportional to the rate of increase of the mole number per unit mass per unit time. The precise definition of F is embodied in the defining formula

$$W_2 = \rho \, m_2 \eta_2 \, F \tag{3.3a}$$

Clearly, by (2.14), we have also the similar formula

$$W_1 = \rho \, m_1 \eta_1 \, F \tag{3.3b}$$

The Law of Mass Action states that if (3.1) represents an elementary reaction

$$F = \frac{1}{\rho} \left[k_f \, n_1^{-\eta_1} - k_b \, n_2^{\eta_2} \right] \tag{3.4a}$$

where k_f and k_b are known functions of temperature. If terms of ξ, ρ and T we have

$$F = \frac{k_f(T)}{\rho} \left[\frac{\rho(Y_{10} + m_1 \eta_1 \xi)}{m_1} \right]^{-\eta_1} - \frac{k_b(T)}{\rho} \left[\frac{\rho(Y_{20} + m_2 \eta_2 \xi)}{m_2} \right]^{\eta_2} \tag{3.4b}$$

In a unimolecular reversible reaction we have:

$$F = k_f(T) \left[\frac{1}{m_1} \left(Y_{10} - \frac{Y_{20}}{K(T)} \right) - \xi \left(1 + \frac{1}{K(T)} \right) \right] \tag{3.4c}$$

where $K(T) = k_f / k_b$ is the equilibrium constant.

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When there are only two species, (2.12) and (2.12) can be combined to give the simple relation

$$\vec{V}_i = -\mathcal{D} \; \frac{\nabla Y_i}{Y_i} \tag{3.5}$$

where $\mathcal{D} = D_{12} = D_{21}$. Introducing (3.5), (3.3), (3.2) into the Mass-Balance Equation for the species (i.e., eq. 2.11), we obtain the simple equation

$$\rho \; \frac{D\xi}{Dt} - \nabla \cdot (\rho \mathcal{D} \xi) = \rho F \tag{3.6}$$

We next consider the energy equation for the case $N=2$. Let us substitute (2.5) and (2.19) into (2.17). Introducing the specific heat at constant pressure C_{Pi} of the species M_i and making use of (2.11) we obtain an alternate form of the Energy Equation

$$\sum_{i=1}^{2} \rho Y_i C_{Pi} \left(\frac{DT}{Dt} + \vec{V}_i \cdot \nabla T \right) - \frac{Dp}{Dt} = \nabla \cdot (\lambda \nabla T) + \Phi - \sum_{i=1}^{2} w_i H_i \tag{3.7}$$

Let us now introduce the simplifying assumption that C_{Pi} can be approximated by a mean C_P which is the same for both species over the range of temperature involved. This implies that

$$H_i = C_P (T - T_0) + H_{i0} \tag{3.8}$$

Eq. (3.7) can then be reduced to the simple form:

$$\rho C_P \frac{DT}{Dt} - \frac{Dp}{Dt} = \nabla \cdot (\lambda \nabla T) + \Phi + \rho A F \tag{3.9}$$

where

$$A = m_2 \eta_1 (H_{10} - H_{20}) \tag{3.10}$$

Finally, by substituting (3.2) in (2.10), the equation of state assumes the simple form

$$p = \rho R_0 T \left[1 + \frac{\eta_1 + \eta_2}{R_0} \xi \right] \tag{3.11}$$

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where

$$R_o = R \sum_{i=1}^{2} \frac{Y_{io}}{m_i} \tag{3.12}$$

is the gas constant of the unburned mixture.

Summarizing, in the case of a single reaction involving two species, there are five variables which describe the state of the fluid: $p, \rho, T, \vec{u}$ and ξ These variables are governed by the five equations: (2.10), (3.6), (2.15), (3.9) and (3.11). With a set of properly imposed boundary conditions, the five equations determine completely the five unknowns.

4. <u>Laminar Flame</u>   We consider a low speed steady one-dimensional flow with a rapid change of temperature occuring in a narrow zone (width L) in which most of the chemical reactions occur. By low speed it is meant that the Mach number is small in comparison to unity. Such a flow field describes the change of state in a plane laminar flame. Once we have chosen the solution for such a flow, we will be interested in its stability. We will assume a small perturbation of the one-dimensional flow and see if there is amplification or decay. The problem will be supposed two-dimensional, but the perturbation being slight, the governing equations may be linearized.

5. <u>One-Dimensional Steady Flow</u>   The equations governing such a flow are:

$$\rho_o u_o \frac{du_o}{dx} = - \frac{dp_o}{dx} + \frac{4}{3}\mu \frac{d^2 u_o}{dx^2} \tag{5.1}$$

$$\rho_o u_o = \text{constant} \tag{5.2}$$

$$\rho_o u_o C_P \frac{dT_o}{dx} = \frac{d}{dx}\left(\lambda \frac{dT_o}{dx}\right) + u_o \frac{dp_o}{dx} + \rho_o A F_o + \frac{4}{3}\mu \left(\frac{du_o}{dx}\right)^2 \tag{5.3}$$

$$\rho_o u_o \frac{d\xi_o}{dx} = \frac{d}{dx}\left(\mathcal{D} \rho_o \frac{d\xi_o}{dx}\right) + \rho_o F_o \tag{5.4}$$

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In order to proceed further we must specify the chemical reaction taking place in the flow. For our purpose, it is as general and more convenient to specify an adequate temperature distribution and then deduce the production term from (5.3). Several such temperature distributions might be convenient. Thus, we may have

$$T_o = T_{o\infty} \left[1 + \zeta_{oo} e^{x/L} (1 + e^{x/L})^{-1} \right] \tag{5.5}$$

This is shown in Fig. 1. Or, we might also take a linear distribution of temperature inside the flame, although such a distribution is certainly unrealistic at the limits of the flame. (See Fig. 2) From experimental data, it can be concluded that the Reynolds number of the flow based on the "flame width" is of the order of unity. This can also be seen from (5.3) if one assumes that $\rho_o C_p u_o \frac{dT_o}{dx}$ and $\lambda \frac{d^2 T_o}{dx^2}$ are of the same order, which is certainly the general case. If L is the thickness of the flame, then $|\frac{dA}{dx}| \sim A/L$ and $\rho_o u_o C_p \sim \frac{\lambda}{L}$ That is, $\mathcal{R}e \, P_r \sim 1$ where

$$\mathcal{R}e = \frac{u_{o\infty} L \rho_{o\infty}}{\mu_{o\infty}} \tag{5.6a}$$

$$P_r = \frac{\mu_{o\infty} C_p}{\lambda_{o\infty}} \tag{5.6b}$$

Now for an ordinary gas, $P_r \sim 1$ so that

$$\mathcal{R}e \sim 1 \tag{5.6}$$

Eq. (2.1) then show that $\frac{dp_o}{dx} \sim \rho_o u_o \frac{du_o}{dx}$ or

$$p = p_{o\infty} \left[1 + O(M^2) \right] \tag{5.6}$$

Since the Mach number of a flame is small in comparison with unity, it follows from (5.6) that the pressure is constant throughout the flow. Consequently,

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the pressure and viscous terms in (2.3) can be neglected in comparison with the conduction term. We have them

$$\rho_c u_c C_p \frac{dT_0}{dx} = \frac{d}{dx}\left(\lambda \frac{dT_0}{dx}\right) + \rho_0 A F_0 \tag{5.7}$$

which is of the same form as (2.4) with T replaced by $\frac{A}{C_p}\xi$. If the Lewis number,

$$Le = \frac{\mathcal{D}\rho_0 C_p}{\lambda}, \tag{5.8}$$

is chosen to be equal to one, it follows from (2.4) and (5.7) that $\xi_0 = \frac{C_p}{A}T_0 +$ a constant, which can be determined by the condition at infinity. Thus, as $x \to -\infty$, $T_0 \to T_{0\infty}$ and $\xi_c \to 0$. Hence

$$\xi_0 = \frac{C_p}{A}(T_0 - T_{0\infty}) \tag{5.9}$$

In particular, if the distribution (5.5) is valid

$$\xi_0 = \frac{C_p}{A}T_{0\infty}\zeta_{\infty}e^{x/L}(1+e^{x/L})^{-1} \tag{5.10}$$

From the equation state (assuming a uni-molecular reaction),

$$\rho_0 = \frac{\rho_{0\infty}}{T_0}T_{0\infty} \tag{5.11}$$

so that

$$\rho_0 = \rho_{0\infty}\left[1 + \zeta_{\infty}e^{x/L}(1+e^{x/L})^{-1}\right]^{-1} \tag{5.11}$$

for the distribution (5.5). The velocity distribution is given by (5.2):

$$u_0 = \frac{u_{0\infty}}{T_{0\infty}}T_0 \tag{5.12}$$

Finally F_0 is given by (5.4) or by (5.7)

6. <u>Some General Remarks on The Perturbed Flow</u>   The perturbed flow is obviously governed by linear equations. Since the unperturbed field is one-dimensional,

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the coefficients in these equations will depend only on x and not on y, z and t. The stability of flame can either be treated as an initial-value problem or, more simply, by a formal analysis in which one harmonic component is considered. Without any loss of generality we can consider the solution as independent of z. Furthermore, since the coefficients of the equations are independent of y and t we may assume, in a formal analysis, a dependence in y and t of the form $e^{iky + \omega t}$ where k is the wave number. In other words, all the perturbed quantities will be of the form $f(x) e^{iky + \omega t}$.

Let us consider the points of the flame at a certain temperature. It can be easily seen that they will be distributed on a line

$$x = \text{constant} + D e^{iky + \omega t} \tag{6.1}$$

where D and the constant are functions of temperature. If ζ denote the actual temperature; T_0 the undisturbed temperature and T the perturbed temperature,

$$\zeta = T_0 + T \tag{6.2}$$

Let x_c be a point on the constant-temperature surface T. At a later instant, it will be located at the point x where

$$\zeta(x) = T_0(x_0) \tag{6.3}$$

For small disturbances $x \sim x_0$ and we have

$$\zeta(x) = \zeta(x_0) + (x - x_0) \zeta'(x_0) \tag{6.4}$$

Solving for x and making use of (6.2) and (6.3), we have

$$x = x_0 - \frac{T(x_0)}{\zeta'(x_0)} \tag{6.5}$$

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But $T(x_o) = \int(x_o) e^{iky+\omega t}$ and from (6.3) we can express x_o as a function of ζ. Hence, (6.5) can be written as

$$x = x_o(\zeta) + G(\zeta) e^{iky+\omega t} \tag{6.6}$$

which proves the statement (6.1)

The D at a certain reference temperature represents the amplitude of the disturbance. For linearization to be possible, we must have

$$D << k^{-1} \tag{6.7}$$

From the kinetic theory, it is known that

$$\mu \sim \rho a \bar{\lambda} \tag{6.8}$$

where $\bar{\lambda}$ is the mean free path and a, the sound speed. Now $R_e \sim 1$, therefore

$$L \sim \frac{\bar{\lambda}}{M} \tag{6.9}$$

But our equations are valid only if we deal with a continuous flow; that is, all distances entering the problem must be much greater than $\bar{\lambda}$. Eq. (7.9) shows that $L >> \bar{\lambda}$, if $M << 1$, which is our case. But it also shows that no distances can be allowed to be of an order smaller than L. In particular, D is of order L or greater, i.e., the flame displacement is at least of the same order as the thickness of flame. Consequently, it is not possible to linearize the equations inside the flame using x as a variable. One way to overcome this difficulty is to use the temperature as the independent variable instead of x. This is particular suitable since the undistrubed field is also given in terms of the temperature.

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7. Transformation of Variables We now introduce

$$T = T(x, y, t), \qquad \eta = y, \qquad \tau = t \qquad (7.1)$$

as our new independent variable. With this choice x must be regarded as a dependent variable:

$$x = x(T, \eta, \tau) \qquad (7.2)$$

From

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial t} dt \qquad (7.3)$$

we get our rule of differentiation:

$$\frac{\partial}{\partial x} = \left(\frac{\partial x}{\partial T}\right)^{-1} \frac{\partial}{\partial T}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} - \left(\frac{\partial x}{\partial \eta}\right)\left(\frac{\partial x}{\partial T}\right)^{-1} \frac{\partial}{\partial T} \qquad (7.4)$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \left(\frac{\partial x}{\partial \tau}\right)\left(\frac{\partial x}{\partial T}\right)^{-1} \frac{\partial}{\partial T}$$

It is interesting to compare the merits of using x and T as the independent variables. With x as the independent variable, the inside of the flame is very narrow compared with the rest of the flow, and, in fact, as we have seen, it does not allow us to study the inside of the flame by linearization. However the solution given by the equations using x as independent variable is still valid outside the flame. On the other hand if T is used as the independent variable, the flow field outside of the flame becomes the narrow region and our equations will be well adapted to study the inside of the flame.

Let us divide the flow in three regions defined as follows (see $fig 1$)

$$\begin{cases} \text{upstream:} & T - T_{o\infty} \ll T_{o\infty} \\[2ex] \text{flame:} & T - T_{o\infty} \sim T_{o\infty} \\[2ex] \text{downstream:} & T - T_{o\infty}(1 + \zeta_{oo}) \ll T_{o\infty} \end{cases} \qquad (7.5)$$

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The limits between the zones are not well defined, but from (5.5)

$$\frac{dx_0}{d\tau} = - \frac{T_{o\infty}\zeta_{oo}L}{[\tau - T_{o\infty}][\tau - T_{o\infty}(1+\zeta_{oo})]} \tag{7.6}$$

so that we can also say:

$$\begin{cases} \text{upstream:} & \frac{dx_0}{d\tau} \sim \frac{L}{\tau - T_{o\infty}} \gg \frac{L}{T_{o\infty}} \\[2ex] \text{flame:} & \frac{dx_0}{d\tau} \sim \frac{L}{T_{o\infty}} \\[2ex] \text{downstream:} & \frac{dx_c}{d\tau} \sim \frac{L}{\tau - T_{o\infty}(1+\zeta_{oo})} \gg \frac{L}{T_{o\infty}} \end{cases} \tag{7.7}$$

Although this definition is equivalent to the previous one if the distribution (5.5) is valid, it is nevertheless preferable, since physically the temperature distribution must be continuous, but the condition of continuity is less stringent upon the derivative $\frac{dx_0}{d\tau}$ which can vary fast from a near infinite value outside the flame to a finite value inside.

8. Continuity Equation   The continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{8.1}$$

expressed in terms of the new variables is

$$\frac{\partial \rho}{\partial \tau}\frac{\partial x}{\partial \tau} - \frac{\partial \rho}{\partial \tau}\frac{\partial x}{\partial \tau} + \frac{\partial \rho u}{\partial \tau} + \frac{\partial \rho v}{\partial \eta}\frac{\partial x}{\partial \tau} - \frac{\partial \rho v}{\partial \tau}\frac{\partial x}{\partial \eta} = 0 \tag{8.2}$$

and using the linearization process:

$$\frac{\partial \rho}{\partial \tau}\frac{dx_0}{d\tau} - \frac{df_c}{d\tau}\frac{\partial x}{\partial \tau} + \frac{\partial}{\partial \tau}(\rho_0 u + \rho u_0) + \rho_0 \frac{\partial v}{\partial \eta}\frac{dx_0}{d\tau} = 0 \tag{8.3}$$

9. Momentum Equation: The two momentum equations are

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2}\right] \tag{9.1}$$

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$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{4}{3}\frac{\partial^2 u}{\partial x^2} + \frac{1}{3}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x^2}\right]$$

In terms of the new variables, we have

$$\rho\left[\frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \tau} - \frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \tau} + u\frac{\partial u}{\partial \tau} + v\frac{\partial u}{\partial \eta}\frac{\partial x}{\partial \tau} - v\frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \eta}\right] = -\frac{\partial p}{\partial \tau} + V.T.$$

$$\rho\left[\frac{\partial v}{\partial \tau}\frac{\partial x}{\partial \tau} - \frac{\partial v}{\partial \tau}\frac{\partial x}{\partial \tau} + u\frac{\partial v}{\partial \tau} + v\frac{\partial v}{\partial \eta}\frac{\partial x}{\partial \tau} - v\frac{\partial v}{\partial \tau}\frac{\partial y}{\partial \eta}\right] = -\frac{\partial p}{\partial \eta}\frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial \tau}\frac{\partial x}{\partial \eta} + V.T. \tag{9.2}$$

where "V.T." stands for the viscous term. These terms are calculated below:

$$\left(\frac{\partial x}{\partial \tau}\right)^3 \frac{\partial^2 u}{\partial x^2} = \frac{\partial x}{\partial \tau}\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial u}{\partial \tau}\frac{\partial^2 x}{\partial \tau^2}$$

$$\left(\frac{\partial x}{\partial \tau}\right)^3 \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial x}{\partial \tau}\right)^2\frac{\partial^2 u}{\partial \tau \partial \eta} + \frac{\partial x}{\partial \tau^2}\frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \eta} - \frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \tau \partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial^2 u}{\partial \tau^2}\frac{\partial x}{\partial \tau}$$

$$\left(\frac{\partial x}{\partial \tau}\right)^3 \frac{\partial^2 u}{\partial y^2} = -\frac{\partial x}{\partial \eta}\frac{\partial^2 u}{\partial \tau \partial \eta}\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial x}{\partial \eta}\right)^2\frac{\partial^2 u}{\partial \tau^2}\frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial \tau}\frac{\partial x}{\partial \eta}\frac{\partial u}{\partial \tau}\frac{\partial x}{\partial \tau \partial \eta}$$
$$-\left(\frac{\partial x}{\partial \eta}\right)^2\frac{\partial}{\partial \tau}\left(\frac{\partial^2 x}{\partial \tau^2}\right) + \left(\frac{\partial x}{\partial \tau}\right)^2\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial x}{\partial \eta}\left(\frac{\partial x}{\partial \tau}\right)^2\frac{\partial u}{\partial \tau \partial \eta} - \frac{\partial^2 x}{\partial \eta}\frac{\partial u}{\partial \tau}\left(\frac{\partial x}{\partial \tau}\right)^2 \tag{9.3}$$

etc.

Applying the linearization procedure, we obtain finally the perturbed momentum equations:

$$\left(\frac{dx_0}{d\tau}\right)^2\left[\rho u_0\frac{du_r}{d\tau} + \rho_0\left(\frac{\partial u}{\partial \tau}\frac{dx_0}{d\tau} - \frac{du_r}{d\tau}\frac{\partial x}{\partial \tau} + u_0\frac{\partial u}{\partial \tau} + u\frac{du_r}{d\tau}\right) + \frac{\partial p}{\partial \tau}\right]$$

$$= 2\frac{\partial x}{\partial \tau}\frac{dx_0}{d\tau}\left[-\frac{dp_0}{d\tau} - \rho_0 u_0\frac{du_0}{d\tau}\right]$$

$$+ \mu\left[\frac{4}{3}\frac{dx_0}{d\tau}\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial u}{\partial \tau}\frac{d^2 x_0}{d\tau^2} - \frac{du_r}{d\tau}\frac{\partial^2 x}{\partial \tau^2} + \left(\frac{dx_0}{d\tau}\right)^3\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial^2 x}{\partial \eta^2}\frac{du_r}{d\tau}\left(\frac{dx_0}{d\tau}\right)^2 + \frac{1}{3}\left(\frac{dx_0}{d\tau}\right)^2\frac{\partial^2 v}{\partial \tau \partial \eta}\right]$$

$$\tag{9.4}$$

$$\left(\frac{dx_0}{d\tau}\right)^2\left[\rho_0\left(\frac{\partial v}{\partial \tau}\frac{dx_0}{d\tau} + u_0\frac{\partial v}{\partial \tau}\right) + \frac{\partial p}{\partial \eta}\frac{dx_0}{d\tau} - \frac{dp_0}{d\tau}\frac{\partial x}{\partial \eta}\right]$$

$$= \mu\left[\frac{4}{3}\left(\frac{dx_0}{d\tau}\right)^3\frac{\partial^2 v}{\partial \eta^2} + \frac{dx_0}{d\tau}\frac{\partial^2 v}{\partial \tau^2} - \frac{dx_0}{d\tau}\frac{\partial v}{\partial \tau} + \frac{1}{3}\left(\frac{dx_0}{d\tau}\right)^2\frac{\partial^2 u}{\partial \tau \partial \eta} + \frac{1}{3}\frac{d^2 x_0}{d\tau^2}\frac{du_0}{d\tau}\frac{\partial x}{\partial \eta}\right.$$
$$\left. - \frac{1}{3}\frac{\partial^2 x}{\partial \eta \partial \tau}\frac{du_0}{d\tau}\frac{dx_0}{d\tau}\right]$$

where use has been made of the relation $u_0 = \frac{u_{0\infty}}{T_{0\infty}}T_0$

10. **Diffusion Equation** The diffusion equation for a two component mixture with Lewis number equal to unity is

$$\rho\left[\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y}\right] - \nabla\cdot\frac{\lambda}{c_p}\nabla\xi) = \rho\bar{F} \tag{10.1}$$

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In terms of the new variables we have

$$\rho\left[\frac{\partial\xi}{\partial\tau}\frac{\partial x}{\partial\tau} - \frac{\partial\xi}{\partial\tau}\frac{\partial y}{\partial\tau} + u\frac{\partial\xi}{\partial\tau} + v\frac{\partial\xi}{\partial\eta}\frac{\partial x}{\partial\tau} - v\frac{\partial\xi}{\partial\tau}\frac{\partial x}{\partial\eta}\right] - \rho F\frac{\partial x}{\partial\tau}$$
$$= \frac{1}{c_p}\frac{\partial}{\partial\tau}\left[\lambda\frac{\partial\xi}{\partial\tau}/\frac{\partial x}{\partial\tau}\right] + \frac{\lambda}{c_p}\left[\frac{\partial x}{\partial\tau}\frac{\partial}{\partial\eta} - \frac{\partial x}{\partial\eta}\frac{\partial}{\partial\tau}\right]\left[\frac{\partial\xi}{\partial\eta} - \frac{\partial x}{\partial\eta}\left(\frac{\partial x}{\partial\tau}\right)^{-1}\frac{\partial\xi}{\partial\tau}\right] \tag{10.2}$$

$F(\rho,\tau,\xi)$ is a known function of ρ, τ and ξ.

When the flow field is perturbed slightly

$$F = F_0 + F_1 \tag{10.3}$$
$$F_1 = \rho\left(\frac{\partial F}{\partial\rho}\right)_0 + \bar{s}\left(\frac{\partial F}{\partial\xi}\right)_0$$

where $F_0 = F(\rho_0,\tau,\xi_0)$ is the production rate at T in the undisturbed flow.

Remembering that $\frac{d^2\xi_0}{d\tau_0^2} = 0$ the linearized diffusion equation assumes the form:

$$\left(\frac{dx_0}{d\tau}\right)^2\left[\rho u_0\frac{d\xi_0}{d\tau} + \rho_0\left(\frac{\partial\xi}{\partial\tau}\frac{dx_0}{d\tau} - \frac{d\xi_0}{d\tau}\frac{\partial x}{\partial\tau} + u_0\frac{\partial\xi}{\partial\tau} + u\frac{d\xi_0}{d\tau}\right) - \rho_0\frac{dx_0}{d\tau}F_1 - \rho F_0\frac{dx_0}{d\tau} - 3\rho_0 F_0\frac{\partial x}{\partial\tau}\right]$$
$$+ 2\frac{\partial x}{\partial\tau}\frac{dx_r}{d\tau}\rho_0 u_0\frac{d\xi_0}{d\tau} \tag{10.4}$$
$$= \frac{\lambda}{c_p}\left[\frac{\partial^2\xi}{\partial\tau^2}\frac{dx_r}{d\tau} - \frac{\partial^2 x}{\partial\tau^2}\frac{d\xi_0}{d\tau} - \frac{d^2 x_0}{d\tau^2}\frac{\partial\xi}{\partial\tau} + \left(\frac{dx_0}{d\tau}\right)^3\frac{\partial^2\xi}{\partial\eta^2} - \left(\frac{dx_r}{d\tau}\right)^2\frac{\partial^2 x}{\partial\eta^2}\frac{d\xi_0}{d\tau_0}\right]$$

11. **Energy Equation:** The Energy Equation:

$$\rho c_p\frac{DT}{Dt} = \nabla\cdot(\lambda\nabla T) + \frac{Dp}{Dt} + \Phi + \rho A F \tag{11.1}$$

expressed in terms of the new variable is

$$\left(\frac{\partial x}{\partial\tau}\right)^2\left[\rho c_r\left(u - \frac{\partial x}{\partial\tau} + v\frac{\partial x}{\partial\eta}\right) - \frac{\partial p}{\partial\tau}\frac{\partial x}{\partial\tau} - u\frac{\partial p}{\partial\tau} - v\frac{\partial p}{\partial\eta}\frac{\partial x}{\partial\tau} + v\frac{\partial p}{\partial\tau}\frac{\partial x}{\partial\eta}\right]$$
$$= \lambda\left[-\frac{\partial^2 x}{\partial\tau^2} + \frac{\partial x}{\partial\tau}\frac{\partial^2 x}{\partial\tau\partial\eta}\frac{\partial x}{\partial\eta} - \left(\frac{\partial x}{\partial\tau}\right)^2\frac{\partial^2 x}{\partial\eta^2} - \frac{\partial^2 x}{\partial\tau^2}\left(\frac{\partial x}{\partial\eta}\right)^2 + \frac{\partial^2 x}{\partial\tau\partial\eta}\frac{\partial x}{\partial\eta}\frac{\partial x}{\partial\tau}\right] \tag{11.2}$$
$$+ \left(\frac{\partial x}{\partial\tau}\right)^3\left[\Phi + \rho A F\right]$$

Now, the dissipation function Φ is given by

$$\Phi/\mu = \frac{4}{3}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] - \frac{4}{3}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \tag{11.3}$$

Thus it contains terms of the following types:

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} ; \left(\frac{\partial u}{\partial x}\right)^2 ; \left(\frac{\partial u}{\partial y}\right)^2 ; \left(\frac{\partial v}{\partial x}\right)^2 ; \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} ; \left(\frac{\partial v}{\partial y}\right)^2$$

Obviously in the linearisation process only the first two terms will give any

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contribution.

Thus we may write

$$\left(\frac{\partial x}{\partial T}\right)^3 \Phi = \phi_o + \phi_1 \tag{11.4}$$

where ϕ_o is the steady dissipation function, and ϕ_1 the first order dissipation function:

$$\phi_o = \mu \left[\frac{4}{3} \frac{dx_o}{dT} \left(\frac{du_o}{dT}\right)^2 \right]$$

$$\phi_1 = \mu \left[-\frac{4}{3} \frac{du_o}{dT} \left(\frac{dx_o}{dT}\right)^2 \frac{\partial v}{\partial \eta} + \frac{8}{3} \frac{du_o}{dT} \frac{dx_o}{dT} \frac{\partial u}{\partial T} + \frac{4}{3} \frac{\partial x}{\partial T} \left(\frac{du_o}{dT}\right)^2 \right] \tag{11.5}$$

Applying the linearization procedure to (11.2), we obtain the energy equation:

$$2 \frac{\partial x}{\partial T} \frac{dx_o}{dT} \left[\rho_o C_p u_o - u_o \frac{dp_o}{dT} \right] + \left(\frac{dx_o}{dT}\right)^2 \left[\rho C_p u_o + \beta C_p (u - \frac{\partial x}{\partial t}) - \frac{\partial p}{\partial t} \frac{dx_o}{dT} - u \frac{dp_o}{dT} - u_o \frac{\partial p}{\partial T} \right]$$

$$= -\lambda \left[\frac{\partial^2 x}{\partial T^2} + \frac{\partial^2 x}{\partial \eta^2} \left(\frac{dx_o}{dT}\right)^2 \right] + \phi_1 + 3 \frac{\partial x}{\partial T} \left(\frac{dx_o}{dT}\right)^3 \rho_o A F_o + A \left(\frac{dx_o}{dT}\right)^3 \left[\rho F_o + \rho_o F_1 \right] \tag{11.6}$$

12. <u>Independence Principle and Justification of Landau's</u> <u>Model</u>   Since the order of magnitude of each physical quantity can be easily estimated, it is possible to rewrite our equations such that each term in its nondimensional form will be of the order of unity.   It is convenient to introduce the following parameter:

$$\varepsilon = D k \tag{12.1}$$

ε is a measure of the intensity of the distrubance since it is the ratio of flame displacement to the wave length of the distrubance. In our problem, we have three length scales: k^{-1} , D and L. In the y, η -direction we must obviously use k^{-1} as the nondimensionalizing factor; in the x-direction we should make a distinction between the perturbed and unperturbed quantities:

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we must use L to non-dimensionalize x_o and D to non-dimensionalize x. In
order to facilitate the identification of the terms we will keep the same
notation whether the quantity has a dimension or not. Table I will be use-
ful in establishing the relation between the dimensional and non-dimensional
form of the same variable. When the governing equations are written in the
non-dimensional form, the following dimensionless numbers appear in the

Table I

Variable Quantities	To Be Non-dimensionalized By
u_o	$u_{o\infty}$
u, v, ω	$\varepsilon u_{o\infty}$
x_o	L
x	εD
η	k^{-1}
τ	$(u_o k)^{-1}$
ξ_o	$C_p T_{o\infty} A^{-1}$
ξ	$\varepsilon C_p T_{o\infty} A^{-1}$
F_o	$u_{o\infty}/ Lm_1\eta_1$
F_1	$\varepsilon u_{o\infty}/ Lm_1\eta_1$
p_o	$p_{o\infty}$
p	$\varepsilon p_{o\infty} \gamma M^2$
ρ_o	$\rho_{o\infty}$
ρ	$\varepsilon \rho_{o\infty}$
T	$T_{o\infty}$

equations: Mach number, Prandtl number, Reynolds number, (Lewis number, which
has been taken equal to unity). In addition, we have the dimensionless number

$$\mathcal{L} = (kL)^{-1} \qquad (12.2)$$

which is the ratio of flame thickness to the wave length of the disturbance, and the dimensionless number $W = A/m_1 \eta_1 C_p T_{\infty}$ which represents the ratio of heat of reaction to the heat capacity of the mixture. In the following, we shall neglect terms of the order M^2 in comparison with unity. In terms of the non-dimensionalized variables, summarized in Table I, the governing equations assume the following form:

$$\frac{d\rho_o}{dT}\left(u - \frac{\partial x}{\partial z}\right) + \rho_o \frac{\partial u}{\partial T} + \frac{\rho_o}{\mathcal{L}} \frac{dx_o}{dT} \frac{\partial v}{\partial \eta} = 0 \tag{12.3}$$

$$\left(\frac{dx_o}{dT}\right)^3 \left[\frac{\partial p}{\partial z} + 2\rho_o \frac{du_o}{dT}\left(u - \frac{\partial x}{\partial z}\right) + \frac{\rho_o}{\mathcal{L}} \frac{dx_o}{dT}\left(\frac{\partial u}{\partial z} - u_o \frac{\partial v}{\partial \eta}\right) \right]$$
$$= \frac{1}{Re}\left[\frac{4}{3} \frac{\partial u}{\partial T} \mathcal{L}\left(\frac{dx_o}{dT}\right)^2 - \frac{4}{3} \frac{\partial u}{\partial T} \frac{dx_o}{dT} \frac{d^2x_o}{dT^2} - \frac{4}{3} \mathcal{L} \frac{du_o}{dT} \frac{dx_o}{dT} \frac{\partial x}{\partial T} + \frac{8}{3} \mathcal{L} \frac{du_o}{dT} \frac{d^2x_o}{dT^2} \frac{\partial x}{\partial T} \right.$$
$$\left. + \frac{1}{3\mathcal{L}}\left(\frac{dx_o}{dT}\right)^3 \frac{\partial^2 v}{\partial T \partial \eta} + \frac{1}{\mathcal{L}} \mathcal{L}\left(\frac{dx_o}{dT}\right)^3 \frac{\partial^2 u}{\partial \eta^2} - \frac{1}{\mathcal{L}} \frac{du_o}{dT}\left(\frac{dx_o}{dT}\right)^3 \frac{\partial^2 x}{\partial \eta^2} \right] \tag{12.4}$$

$$\left(\frac{dx_o}{dT}\right)^2 \left[\frac{\partial v}{\partial T} + \frac{\partial x}{\partial \eta} \frac{du_o}{dT} + \frac{1}{\mathcal{L}}\left(\rho_o \frac{\partial v}{\partial z} + \frac{\partial p}{\partial \eta}\right)\frac{dx_o}{dT} \right]$$
$$= \frac{1}{Re}\left[\frac{4}{3} \frac{1}{\mathcal{L}^2}\left(\frac{dx_o}{dT}\right)^3 \frac{\partial^2 v}{\partial \eta^2} + \frac{1}{3\mathcal{L}} \frac{\partial^2 u}{\partial T \partial \eta}\left(\frac{dx_o}{dT}\right)^2 - \frac{dx_o}{dT} \frac{du_o}{dT} \frac{\partial x}{\partial \eta} + \frac{\partial v}{\partial T^2} \frac{dx_o}{dT} - \frac{\partial v}{\partial T} \frac{d^2x_o}{dT^2} - \frac{1}{3} \frac{\partial^2 x}{\partial T \partial \eta} \frac{dx_o}{dT} \frac{du_o}{dT} \right] \tag{12.5}$$

$$\left(\frac{dx_o}{dT}\right)^2 \left[\frac{\rho_o}{\mathcal{L}} \frac{\partial \xi}{\partial z} \frac{dx_o}{dT} - \rho_o \frac{d\xi_o}{dT} \frac{\partial x}{\partial z} + \frac{\partial \xi}{\partial T} + \rho_o u \frac{d\xi_o}{dT} - \left(\rho_o \frac{d\xi_o}{dT} F_1 + 3\mathcal{L}\rho_o F_o \frac{\partial x}{\partial z}\right) \right] + 2\mathcal{L} \frac{\partial x}{\partial T} \frac{dx_o}{dT} \frac{d\xi_o}{dT}$$
$$= \frac{1}{Re P_r}\left[\frac{\partial^2 \xi}{\partial T^2} \frac{dx_o}{dT} - \mathcal{L} \frac{\partial x}{\partial T^2} \frac{d\xi_o}{dT} - \frac{d^2x_o}{dT^2} \frac{\partial \xi}{\partial T} - \frac{1}{\mathcal{L}^2} \frac{\partial^2 \xi}{\partial \eta^2}\left(\frac{dx_o}{dT}\right)^3 - \frac{1}{\mathcal{L}}\left(\frac{dx_o}{dT}\right)^2 \frac{\partial^2 x}{\partial \eta^2} \frac{d\xi_o}{dT} \right] \tag{12.6}$$

$$2\mathcal{L} \frac{\partial x}{\partial T} \frac{dx_o}{dT} + \left(\frac{dx_o}{dT}\right)^2 \left[\rho_o\left(u - \frac{\partial x}{\partial z}\right)\right]$$
$$= -\frac{\mathcal{L}}{Re P_r}\left[\frac{\partial^2 x}{\partial T^2} + \frac{1}{\mathcal{L}^2}\left(\frac{dx_o}{dT}\right)^2 \frac{\partial^2 x}{\partial \eta^2} \right] + \left(\frac{dx_o}{dT}\right)^2 \left[\frac{dx_o}{dT} \rho_o F_1 + 3\rho_o F_o \mathcal{L} \frac{\partial x}{\partial T} \right] W \tag{12.7}$$

$$\rho = O(M^2) \tag{12.8}$$

Eq. (12.3) is the continuity equation; (12.4) and (12.5) are the momentum

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equations; (12.6) and (12.7) respectively are the diffusion equation and the

energy equation ; (12.8) is the equation of state, all written in dimension-

less form. Note that (12.8) shows that the compressibility effect is negligible.

The non-dimensional variables p_o , ρ_o , u_o , ξ_o , x_o describing the undis-

turbed flame are related to one another by

$$\frac{du_o}{dT} = \frac{d\xi_o}{dT} = \frac{d}{dT}\left(\frac{1}{\rho_o}\right) = 1 \tag{12.9}$$

$$\frac{1}{\delta M^2}\left(\frac{dx_o}{dT}\right)^2 \frac{dp_o}{dT} = -\frac{du_o}{dT}\left(\frac{dx_o}{dT}\right)^2 - \frac{4}{3}\frac{1}{Re}\frac{du_o}{dT}\frac{d^2x_o}{dT^2} \tag{12.10}$$

Substracting (12.7) from (12.6) and making use of (12.9), we obtain:

$$\left(\frac{dx_o}{dT}\right)^2\left[\frac{\rho_o}{L}\frac{\partial\xi}{\partial\zeta}\frac{dx_o}{\partial T} + \frac{\partial\xi}{\partial T}\right] = \frac{1}{Re\,Pr}\left[\frac{\partial^2\xi}{\partial T^2}\frac{dx_o}{dT} - \frac{d^2x_o}{dT^2}\frac{\partial\xi}{\partial T} - \frac{1}{L^2}\frac{\partial^2\xi}{\partial\eta^2}\left(\frac{dx_o}{dT}\right)^3\right] \tag{12.11}$$

Now the solution of this equation having an exponential dependence on η and

τ which satisfies the homogeneous boundary conditions $\xi = 0$ at the upstream

and downstream infinity is clearly

$$\xi = 0 \tag{12.12}$$

This equation states the familiar fact that, when Lewis number is equal to one,

the composition of the mixture is directly related to the temperature of the

mixture.

It is evident that a flame can be approximated as a surface of discontinuity

only when the wave length of the disturbance is much greater than the thickness

of the flame. Hence, Landau's model is strictly applicable only when $\lambda \to \infty$.

In Landau's model, the flame is considered as a surface of discontinuity across

which there is a discontinuous temperature change and which propagates with a

constant speed relative to the fluid immediately ahead of the surface. If

Landau's phenomenological model gives a correct description of a flame when

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$\mathcal{L} \to \infty$, it must be possible to justify it by using the fluid dynamic equations governing a reactive mixture as the basis of analysis. Furthermore, if Landau's model can be justified, it paves the way for improving the model so that the effect of heat conduction and viscosity may be taken into account.

To justify Landau's model, let us expand the various dependent variables in an inverse power series of the parameter $\mathcal{L}$:

$$x = \bar{x}_0 + \frac{1}{\mathcal{L}} \bar{x}_1 + \cdots$$
$$p = \bar{p}_0 + \frac{1}{\mathcal{L}} \bar{p}_1 + \cdots \qquad (12.13)$$
$$u = \bar{u}_0 + \frac{1}{\mathcal{L}} \bar{u}_1 + \cdots$$

etc. Substitute these expressions into (12.3), (12.4), (12.5), (12.12), (12.7), (12.8) and equate terms of the same degree in $\mathcal{L}$.

First, let us consider terms of order $\mathcal{L}^1$. Eqs. (12.4) and (12.7) yield the following relations:

$$-\frac{dx_0}{dT} \frac{\partial^2 \bar{x}_0}{\partial T^2} + 2\frac{d^2 x_0}{dT^2} \frac{\partial \bar{x}_0}{\partial T} = 0$$

$$\frac{1}{R_e P_r} \frac{\partial^2 \bar{x}_0}{\partial T^2} + 2\frac{dx_0}{dT} \left[1 - \frac{3}{2} F_0 \rho_0 w \frac{dx_0}{dT} \right] \frac{\partial \bar{x}_0}{\partial T} = 0 \qquad (12.14)$$

These equations can be considered as a pair of simultaneous homogeneous algebraic equations governing the two variables: $\frac{\partial^2 \bar{x}_0}{\partial T^2}$ and $\frac{\partial \bar{x}_0}{\partial T}$. Since the determinent of the system is equal to $-\left(\frac{dx_0}{dT}\right)^3 F_0 \rho_0 w$ and is different from zero inside the combustion zone, it follows that

$$\frac{\partial \bar{x}_0}{\partial T} = 0 \qquad (12.15)$$

Consequently, when the wave length of the disturbance is large in comparison with the thickness of the flame so that all terms of order $1/\mathcal{L}$ and higher can be neglected, the thickness of the flame remains constant when it is slightly dis-

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turbed. In other words, the effect of the disturbance is to carry the flame along without changing its structure.

Let us next collect terms of order $\mathcal{L}^{\circ}$.

From (12.4), we find

$$\frac{d\rho_o}{d\tau}\left(\bar{u}_o - \frac{\partial\bar{x}_c}{\partial\tau}\right) + \rho_o\frac{\partial\bar{u}_o}{\partial\tau} = 0 \tag{12.16}$$

Using (12.15), we may rewrite (12.16) as

$$\frac{d}{d\tau}\left[\rho_o\left(\bar{u}_o - \frac{\partial\bar{x}_o}{\partial\tau}\right)\right] = 0$$

or

$$\rho_o\left(\bar{u}_o - \frac{\partial\bar{x}_o}{\partial\tau}\right) = C_1 \tag{12.17}$$

where C_1 is a constant which may, however, be a function of η and τ. Eq. (12.17) expresses the fact that the mass flux through the flame is constant. From (12.6), we find:

$$\left(\frac{dx_o}{d\tau}\right)^2\frac{\partial}{\partial\tau}\left[\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right] = \frac{1}{Re}\left[-\frac{d^2x_o}{d\tau^2}\frac{\partial}{\partial\tau}\left(\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right) + \frac{dx_o}{d\tau}\frac{\partial^2\bar{v}_o}{\partial\tau^2}\right] \tag{12.18}$$

Now, from (12.9), $\frac{du_o}{d\tau} = 1$. This equation can thus be rewritten as

$$\frac{\partial}{\partial\tau}\left[\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right] = \frac{1}{Re}\frac{\partial}{\partial\tau}\left[\left(\frac{dx_o}{d\tau}\right)^{-1}\frac{\partial}{\partial\tau}\left(\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right)\right] \tag{12.19}$$

which may be integrated at once to give

$$\left[\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right] - \frac{1}{Re}\left(\frac{dx_o}{d\tau}\right)'\left[\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right] = C_2 \tag{12.20}$$

The equation has a simple meaning. First of all, we note that $\frac{\partial\bar{x}_o}{\partial\eta}$ is the slope of the constant temperature surface so that $\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}$ represents the tangential velocity of the flow relative to an observer fixed on the constant-temperature surface. The term $\left(\frac{dx_o}{d\tau}\right)^{-1}\left[\bar{v}_o + u_o\frac{\partial\bar{x}_o}{\partial\eta}\right]$ thus represents the velocity gradient at the constant-temperature surface and

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eq. (12.20) states that tangential momentum transport minus the shearing stress is continuous across the flame. If the viscous effect at two sides of the combustion zone is negligible, we have simply the condition of continuity of tangential velocity:

$$\bar{v}_0 + u_0 \frac{\partial \bar{x}_0}{\partial \eta} = C_2 \tag{12.21}$$

We next turn to eq. (12.7). Collecting terms of order $\mathcal{L}^\circ$, we have

$$\left(\frac{dx_0}{d\tau}\right)^2 \rho_0 \left(\bar{u}_0 - \frac{\partial \bar{x}_0}{\partial \tau}\right) = -\frac{1}{R_e P_r} \frac{\partial^2 \bar{x}_1}{\partial \tau^2} - 2\frac{dx_0}{d\tau}\left[1 - \frac{3}{2} F_0 \rho_0 w \frac{dx_0}{d\tau}\right]\frac{\partial \bar{x}_1}{\partial \tau} \tag{12.22}$$

This equation can be interpreted as an equation governing the change in the thickness of flame. For, if eq. (12.17) is substituted into (12.22), we obtain a second order differential equation for $\bar{x}_1$:

$$\frac{1}{R_e P_r} \frac{\partial^2 \bar{x}_1}{\partial \tau^2} + 2\frac{dx_0}{d\tau}\left[1 - \frac{3}{2} F_0 \rho_0 w_0 \frac{d\bar{x}_0}{d\tau}\right]\frac{\partial \bar{x}_1}{\partial \tau} = -c_1 \left(\frac{dx_0}{d\tau}\right)^2 \tag{12.23}$$

Obviously the change in the flame thickness caused by disturbances in the flow must depend on the reaction mechanism and the structure of the flame, as is indeed embodied in the terms F_0 and x_0 in (12.23). This effect will be important only when $\mathcal{L}$ becomes less than unity.

Eq. (12.22) can also be regarded as an equation governing the change of $\left(\bar{u}_0 - \frac{\partial \bar{x}_0}{\partial \tau}\right)$ through the flame. For $\mathcal{L} \gg 1$, we expect that the variation of all quantities of the order of $\mathcal{L}^\circ$ or even of the order $\mathcal{L}^{-1}$ should be independent of the structure $x_0(\tau)$ of the flame. Assuming this "Independence Principle", we conclude immediately from (12.23) that

$$\frac{\partial \bar{x}_1}{\partial \tau} = 0 \tag{12.24}$$

and hence

$$\bar{u}_0 = \frac{\partial \bar{x}_0}{\partial \tau} \tag{12.25}$$

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Equation (12.25) states that the constant-temperature surface moves at a constant velocity, since the perturbed velocity relative to an observer fixed on the constant-temperature surface is $\bar{u}_o - \frac{\partial \bar{x}_o}{\partial \tau}$ which, according to (12.25), is zero. This substantiates Landau's model.

Eq. (12.24) states that, even to the order of $^1/\mathcal{L}$, the flame thickness is independent of the flow disturbances.

If we had applied the Independence Principle to eq. (12.20) or (12.19), we would have obtained directly (12.21).

Finally, let us turn to eq. (12.4). Collecting terms of the order of $\mathcal{L}^0$ and making use of (12.15), (12.24), (12.25), we obtain:

$$\frac{\partial \bar{p}_o}{\partial \tau} = 0 \tag{12.26}$$

That is, the perturbed pressure is continuous across the combustion zone.

Obviously eqs. (12.25), (12.17), (12.21), (12.26) are identical to the four boundary conditions which can be deduced for a surface of discontinuity of Landau's type. This justifies Landau's model for a flame when the Independence Principle is applicable. Of course, this principle itself is a postulate, the correctness of which must be verified by the internal con-sistency and by deductions. However, one thing is clear: when this principle is not applicable, Landau's model is inadequate to describe a flame front. Moreover, it will be seen that important deductions can be made from this hypothesis which defines mathematically meaningful problem and provides physically meaningful results.

13. Effect of Conduction on Flame Stability   To deduce a correction to Landau's model for not-so-large $\mathcal{L}$, we shall compute the terms of order $\mathcal{L}^1$ in the expansion (12.13). We shall first consider the effect of heat conduction on the

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stability; and we shall neglect the effect of viscosity altogether. Collecting terms of the order of $1/\mathcal{L}$ and applying the Independence Principle, the continuity, momentum and energy equations yield the following system of equations:

$$\frac{d}{d\tau}\left[\rho_0\left(\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \tau}\right)\right] = 0 \tag{13.1}$$

$$\frac{\partial \bar{\rho}_1}{\partial \tau} + 2\rho_0 \frac{du_0}{d\tau}\left(\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \tau}\right) = 0 \tag{13.2}$$

$$\frac{\partial \bar{v}_1}{\partial \tau} + \frac{\partial \bar{x}_1}{\partial \eta}\frac{du_0}{d\tau} = 0 \tag{13.3}$$

$$\beta_0\left(\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \tau}\right) = -\frac{1}{R_e P_r}\frac{\partial^2 \bar{x}_0}{\partial \eta^2} \tag{13.4}$$

To be exact, eq. (13.1) is derived from (12.3); (13.2) from (12.4); (13.3) from (12.5) and (13.4) from (12.7). In addition, we have (12.24):

$$\frac{\partial \bar{x}_1}{\partial \tau} = 0 \tag{13.5}$$

Eq. (13.2) and (13.1) may be combined to give

$$\frac{\partial \bar{\rho}_1}{\partial \tau} + 2\frac{\partial \bar{u}_1}{\partial \tau} = 0 \tag{13.6}$$

where use has been made of (13.5) and (12.9). This equation may be used instead of (13.2). Now $\frac{\partial \bar{x}_0}{\partial \tau} = 0$ (see eq. 12.15), it follows that eq. (13.4) implies (13.1). Consequently, eq. (13.1) is not an independent relation and need not be considered further. Finally eq. (13.3) can evidently be written as

$$\frac{\partial}{\partial \tau}\left[\bar{v}_1 + u_0\frac{\partial \bar{x}_1}{\partial \eta}\right] = 0 \tag{13.7}$$

Eqs. (13.4), (13.5), (13.6), (13.7) govern the changes of $\bar{u}_1$, $\bar{v}_1$, $\bar{\rho}_1$, $\bar{x}_1$ in the flame. Now $\bar{u}_0$, $\bar{v}_0$, $\bar{\rho}_0$, $\bar{x}_0$ satisfy eqs. (12.25), (12.21), (12.26)

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and (12.15). Thus, they also satisfy the following equations:

$$\rho_0 \left(\bar{u}_0 - \frac{\partial \bar{x}_0}{\partial \tau} \right) = 0$$

$$\frac{\partial \bar{x}_0}{\partial T} = 0 \tag{13.8}$$

$$\frac{\partial \bar{p}_0}{\partial T} + 2 \frac{\partial \bar{u}_0}{\partial T} = 0$$

$$\frac{\partial}{\partial T} \left[\bar{v}_0 + u_0 \frac{\partial \bar{x}_0}{\partial \eta} \right] = 0$$

If we multiply (13.4), (13.5), (13.6), (13.7) by   and add to the corresponding equations of (13.8), we obtain:

$$\rho_0 \left(u - \frac{\partial x}{\partial \tau} \right) = - \frac{1}{Re \, Pr \, \lambda} \frac{\partial^2 x}{\partial \eta^2} \tag{13.9}$$

$$\frac{\partial x}{\partial T} = 0 \tag{13.10}$$

$$\frac{\partial p}{\partial T} + 2 \frac{\partial u}{\partial T} = 0 \tag{13.11}$$

$$\frac{\partial}{\partial T} \left[v + u_0 \frac{\partial x}{\partial \eta} \right] = 0 \tag{13.12}$$

which reduce to the boundary conditions associated with Landau's model if we let $\lambda \to \infty$. Eq. (13.10) shows that x is independent of T so that the flame thickness is not affected by the disturbances in the flow. Eq. (13.9) shows that the flame speed is affected by the curvature of the flame. The negative sign in (13.9) corresponds to the fact that the flame speed increases when the flame front is concave toward the unburned gas. Finally, (13.11) and (13.12) can be integrated to give the boundary conditions:

$$p' + 2u' = p'' + 2u'' \tag{13.13}$$

$$V_t' = V_t'' \tag{13.14}$$

where the prime and double prime signify the state of flow at two sides of the flame and V_t is the tangential velocity

$$V_t = v + u_0 \frac{\partial x}{\partial \eta} \tag{13.15}$$

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The boundary conditions (13.9) - (13.12) are identical to those obtained by a different method in an earlier analysis by the authors. (ref. 8) There it is shown that these boundary conditions are applicable under a less stringent condition. It is further shown that the conduction effect stabilizes the flame if

$$\frac{1}{P_r R_e \mathcal{L}} \geq \frac{1}{2} \tag{13.16}$$

14. **Effect of Viscosity on Flame Stability**   We next examine the effect of viscosity by the same method. Here we shall ignore the influence of heat conduction. Collecting terms of the order of $1/\mathcal{L}$ and applying the Independence Principle, the continuity, momentum and energy equations yield the following system of equations.

$$\frac{d}{dT}\left[\rho_o\left(\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \tau}\right)\right] = 0 \tag{14.1}$$

$$\frac{\partial \bar{p}_1}{\partial T} + 2\rho_o \frac{du_o}{dT}\left(\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \tau}\right) = \frac{1}{R_e}\left[\frac{1}{3}\frac{\partial^2 \bar{v}_o}{\partial T \partial \eta} - \frac{du_o}{dT}\frac{\partial^2 \bar{x}_2}{\partial \eta^2}\right] \tag{14.2}$$

$$\frac{\partial \bar{v}_1}{\partial T} + \frac{\partial \bar{x}_1}{\partial \eta}\frac{du_o}{dT} = \frac{1}{3}\frac{1}{R_e}\frac{\partial^2 \bar{u}_o}{\partial T \partial \eta} \tag{14.3}$$

$$\bar{u}_1 - \frac{\partial \bar{x}_1}{\partial \eta} = 0 \tag{14.4}$$

In addition, we have

$$\frac{\partial \bar{x}_1}{\partial T} = 0 \tag{14.5}$$

Since (14.4) and (14.5) imply the relation (14.1), the latter need not be considered further. Substituting (14.4) and (12.21) into (14.2), we obtain

$$\frac{\partial \bar{p}_1}{\partial T} = -\frac{4}{3}\frac{1}{R_e}\frac{\partial^2 \bar{x}_o}{\partial \eta^2} \tag{14.6}$$

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Finally, substituting (12.25) and (12.15) into (14.3) we have

$$\frac{\partial}{\partial \tau} \left[\bar{v}_1 + u_0 \frac{\partial \bar{x}_1}{\partial \eta} \right] = 0 \qquad (14.7)$$

where use has been made of (14.5). Eqs. (14.4) - (14.7) govern the changes of $\bar{u}_1$, $\bar{v}_1$, $\bar{p}_1$, $\bar{x}_1$ in the combustion zone. Now $\bar{u}_0$, $\bar{p}_0$, $\bar{v}_0$, $\bar{x}_0$ satisfy eqs. (12.25), (12.21), (12.26) and (12.15) so that

$$\bar{u}_0 - \frac{\partial \bar{x}_0}{\partial \tau} = 0$$

$$\frac{\partial \bar{x}_0}{\partial \tau} = 0 \qquad (14.8)$$

$$\frac{\partial \bar{p}_0}{\partial \tau} = 0$$

$$\frac{\partial}{\partial \tau} \left[\bar{v}_0 + u_0 \frac{\partial \bar{x}_0}{\partial \eta} \right] = 0$$

Let us multiply (14.42) to (14.7) by $\mathcal{L}^{-1}$ and add them to the corresponding equations of (14.8). We obtain:

$$u - \frac{\partial x}{\partial \tau} = 0 \qquad (14.9)$$

$$\frac{\partial x}{\partial \tau} = 0 \qquad (14.10)$$

$$\frac{\partial p}{\partial \tau} = -\frac{4}{3} \frac{1}{Re\mathcal{L}} \frac{\partial^2 x}{\partial \eta^2} \qquad (14.11)$$

$$\frac{\partial}{\partial \tau} \left[v + u_0 \frac{\partial x}{\partial \eta} \right] = 0 \qquad (14.12)$$

These boundary conditions reduce to that associated with Landau's model if we let $\mathcal{L} \to \infty$. They have been applied in ref. 8 to study the effect of viscosity on the stability of flame. It is shown that viscous effect cannot stabilize the flame if $Re\mathcal{L}$ has a value greater than one . Thus, conduction has a greater influence than the viscous effect in stabilizing the flame.
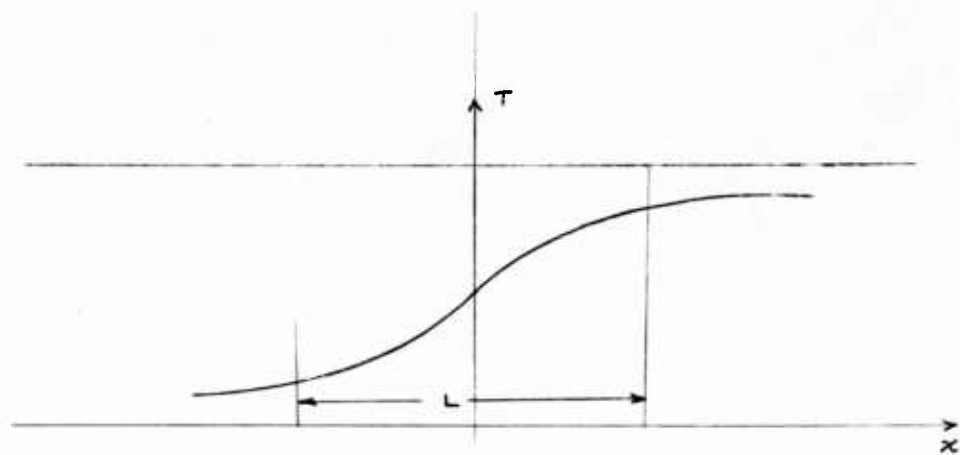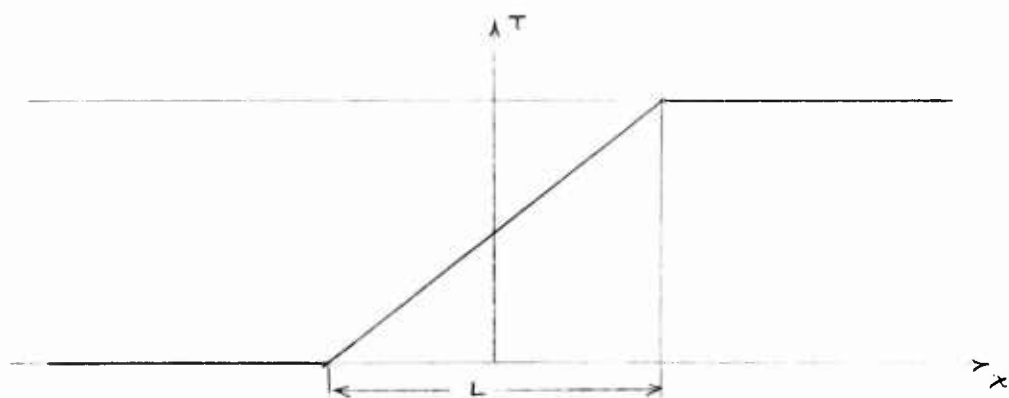
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Fig. 1

Fig. 2

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